The difference between definitions and theorems

It is very common when first taking higher level math (calculus and above) either to be confused by what a definition is versus what a theorem is or to think that it doesn't matter what the difference is.

A definition tells you
what a certain term (a word or a phrase) means,
and when and how to use it.

For example, consider the following definition of "**co-prime**":

DEFINITION: **Two or more integers** $x_1, x_2, x_3, \cdots x_n$ **are co-prime**
if the greatest common factor of $x_1, x_2, x_3, \cdots x_n$ **is** 1.

This definition tells you that

co-prime means that the greatest common factor is 1, and
you use the term "co-prime" when you are talking about two or more integers, and
you use the term "co-prime" by saying that the two or more integers "are co-prime"

Based on the definition, you could say

15 **and** 77 **are co-prime** — **because** 15 **and** 77 **are both integers**
and **the greatest common factor of** 15 **and** 77 **is** 1.
[You can check this by factoring 15 and 77 into primes, and noting that they share no common prime factors.]

You could also say

15, 10 **and** 6 **are co-prime** — **because** 15, 10 **and** 6 **are all integers**
and **the greatest common factor of** 15, 10 **and** 6 **is** 1.
[NOTE: It is not relevant that
the greatest common factor of 15 and 10 is not 1,
the greatest common factor of 15 and 6 is not 1,
and the greatest common factor of 10 and 6 is not 1,
since the definition only refers to the greatest common factor of the entire set of integers at once.]

However,

15 **and** 77 **do NOT have co-prime** — **because the correct usage is "are co-prime"**
[not "have co-prime"].

and

13 **is NOT co-prime** — **because there is only one number mentioned (not two or more)**
[even though 13 is an integer and is prime].

and

14 **and** 77 **are NOT co-prime** — **because the greatest common factor of** 14 **and** 77 **is** 7 **(not** 1**)**
[even though 14 and 77 are both integers].

and

π **and** 7 **are NOT co-prime** — **because** π **is not an integer**
[even though 7 is an integer].

A theorem tells you that,
given a certain set of facts (conditions) about a situation,
another fact (conclusion) must automatically be true.

One purpose of theorems is that
they allow you to claim the conclusion is true
by simply checking that the conditions are true.
(Usually, it is easier to check that the conditions are true, and harder to check that the conclusion is true.)
In other words, theorems are the basis of many shortcuts.

For example, suppose you were trying to determine if 9437 and 8413 are co-prime.
If you only had the definition of co-prime, you would have to
factor both 9437 and 8413,
and check if they share any common factors other than 1.
Factoring of large numbers can take a lot of work.
So, trying to determine if these two numbers are co-prime this way could take a while.

Now, consider the following theorem about co-prime numbers:

THEOREM: **If x and y are integers,**
and none of the prime factors of $x-y$ are factors of y,
then x and y are co-prime.

With this theorem in hand, you could instead check that $9437-8413=1024$,
and since the only prime factor of 1024 is 2,
and 2 is not a factor of 8413,
that automatically means that 9437 and 8413 are co-prime.

Notice how much less work was involved when you used the theorem.
You only had to factor one smaller number (1024) instead of two larger ones (9437 and 8413).

However, notice that if you only knew the theorem and not the definition,
you would only know that 9437 and 8413 are co-prime,
<u>BUT</u> you would still have no idea what it means to say that 9437 and 8413 are co-prime.

But why is it important to know what co-prime means (the definition),
if you already know how to check quickly if two numbers are co-prime (the theorem/shortcut) ?

Because if you don't know what co-prime means,
then you won't be able to come up with new theorems/shortcuts based on two numbers being co-prime.

For example, suppose you need to find the LCD (least common denominator) of the fractions

$$\frac{625}{9437} \text{ and } \frac{256}{8413}$$

Based on your previous experience with fractions, you think that means
you need to factor 9437 and 8413 into primes,
then collect the highest powers of both their prime factors,
and multiply them all together to get the LCD.

However, it turns out that if 9437 and 8413 share no common prime factors,
then the result of factoring and collecting and multiplying described above
gives you that the same result as just multiplying 9437×8413.

In other words, if 9437 and 8413 are co-prime,
then the LCD is just 9437×8413.

And you can check if 9437 and 8413 are co-prime
by simply checking if any of the prime factors of $9437 - 8413$ are factors of 8413.

So, that means you can find the LCD much more quickly
if you know **BOTH**
the definition of co-prime
and the theorem about how to check quickly whether two numbers are co-prime.

So, it is not enough to know only the definitions or only the theorems.
You must know both the definitions and the theorems, and
you must know which are the definitions and which are the theorems.

You must know the definitions, so you know what you and other people mean when you use the term.
And you must know the theorems, so you can check more easily if the term applies to a situation.

And when you know both the definitions and the theorems,
you can combine them together
to get more theorems/shortcuts
that do more work more quickly with less effort.